



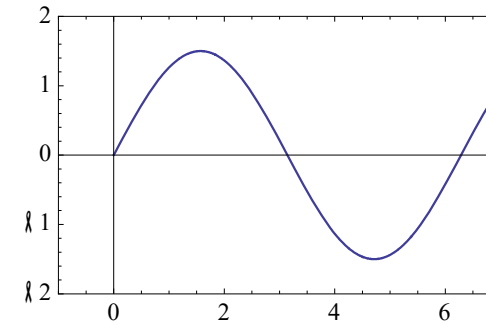
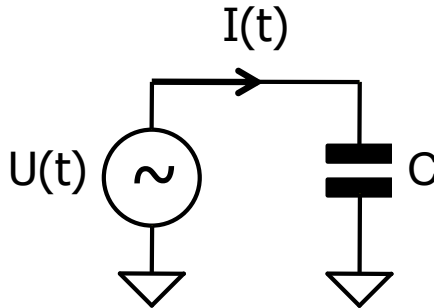
AC BEHAVIOR OF COMPONENTS



AC Behavior of Capacitor

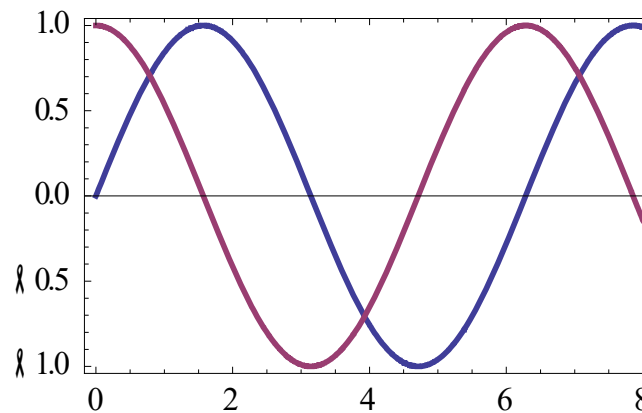
- Consider a capacitor driven by a sine wave voltage:

$$U(t) = U_0 \sin(\omega t + \varphi)$$



- The current: $I(t) = C \frac{dU(t)}{dt} = C U_0 \omega \cos(\omega t + \varphi)$

is shifted by 90° (sin ↔ cos)!



Remember:
Capacitor = Storage
for Charge

Peak current is high for

- large cap
- large voltage
- high frequency... because charge must flow in/out more often in the same time



Complex Impedance

- To simplify our calculations, we would like to extend the relation $R = U/I$ to capacitors, using an **impedance** Z_C .

- In order to get the **phase** right, we use **complex** quantities:

$$U(t) = U_0 \sin(\omega t + \varphi) \quad \rightsquigarrow \quad U_0 \cdot e^{i(\omega t + \varphi)} = U_0 [\cos(\omega t + \varphi) + i \sin(\omega t + \varphi)]$$

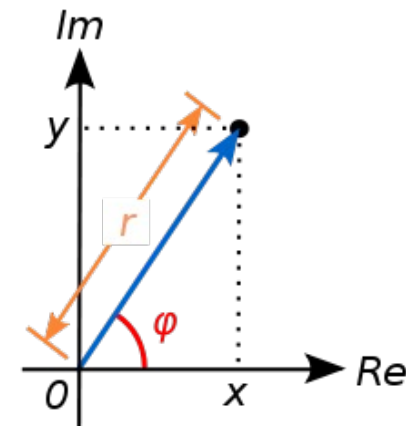
for voltages and currents.

Euler Equation!

- By mixing complex and real parts, we can mix $\sin()$ and $\cos()$ components and therefore influence the phase.

- Note: Often 'j' is used instead of 'i' for the complex unit, because 'i' is used as current symbol...

- Often 's' is used for $i\omega$ (or $j\omega$)

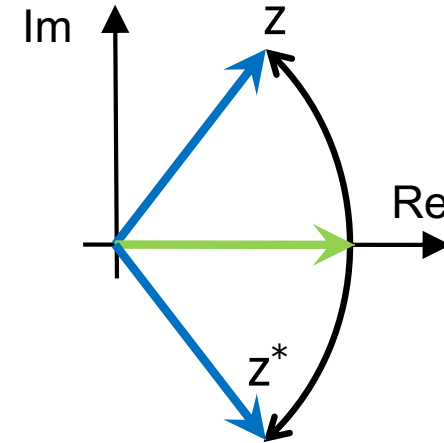




From Complex Values back to Real Quantities

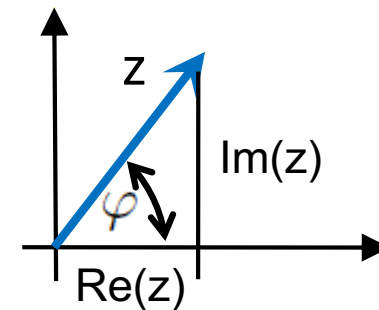
- To find ('back') the **amplitude** of such a complex signal, we calculate the length (**magnitude**) of the complex vector as

$$a = \sqrt{z z^*}$$



- To get the **phase**, we use real and imaginary parts:

$$\varphi = \text{atan} \left(\frac{\text{Im}(z)}{\text{Re}(z)} \right)$$



Note: this simple formula works only in 2 quadrants. You may have to look at the signs of $\text{Re}(z)$ and $\text{Im}(z)$



Hints for Mathematica

- Mathematica knows complex arithmetic
- Useful Functions are **Abs []** and **Arg []**
 - Remember: Imaginary Unit is typed as **ESC i i ESC**
- If you want to simplify expression, Math. has to **know** that expressions like ω , R , C , U are **real** (or even positive)

- This can be done with Assumptions:
- Sometimes **ComplexExpand []** can be used. It assumes all arguments are **real** (but not necessarily > 0):

```
{Abs[i ω], Arg[i ω]} // ComplexExpand
{√ω2, Arg[i ω]}
```

```
$Assumptions = True;
{Abs[i ω], Arg[i ω]}
{Abs[ω], Arg[i ω]}

{Abs[i ω], Arg[i ω]} // FullSimplify
{Abs[ω], Arg[i ω]}

$Assumptions = ω > 0;
{Abs[i ω], Arg[i ω]} // Simplify
{ω, π/2}
```



Complex Impedance of the Capacitor

- We know that

$$I(t) = C \frac{dU(t)}{dt}$$

- With $U(t) = U_0 \cdot e^{i(\omega t + \varphi)}$

we have $I(t) = CU'(t) = C \cdot U_0 \cdot i\omega \cdot e^{i(\omega t + \varphi)}$

- Therefore

$$Z_C = \frac{U(t)}{I(t)} = \frac{1}{i\omega C} = \frac{1}{sC}$$

- Similar:

$$Z_L = i\omega L = sL$$

The impedance of a capacitor becomes very small at high frequencies



Checking this again for a Capacitor

- For an input voltage (sine wave of freq. ω) with phase $\varphi=0$

$$U(t) = U_0 e^{i\omega t}$$

we have

$$I(t) = \frac{U(t)}{Z_C} = U_0 e^{i\omega t} \cdot i\omega C$$

- The amplitude of $I(t)$ is

$$\begin{aligned} |I| &= \sqrt{I(t)I^*(t)} \\ &= \sqrt{U_0 e^{i\omega t} \cdot i\omega C \times U_0 e^{-i\omega t} \cdot (-i)\omega C} \\ &= \sqrt{U_0^2 e^{i\omega t} e^{-i\omega t} \cdot (i\omega C)(-i\omega C)} \\ &= U_0 \omega C \end{aligned}$$

- The phase is (quite complicated...):

$$\varphi_I = \text{atan} \left(\frac{\text{Im}(I)}{\text{Re}(I)} \right) = \text{atan} \left(\frac{U_0 \omega C \cos(\omega t)}{-U_0 \omega C \sin(\omega t)} \right) = -\text{atan} \left(\frac{1}{\tan(\omega t)} \right) = \omega t - \frac{\pi}{2}$$



Simplifying even more

- As we have just seen, the $U(t) = U_0 e^{i\omega t}$ propagates trivially to the output.
- We therefore drop this part in the future and just use '1'!



Recipe to Calculate Transfer Functions

- Replace all component by their complex impedances ($1/(sC)$, sL , R)

- Assume a unit signal of '1' at the input

(in reality it is $U(t) = U_0 e^{i\omega t}$)

- Write down all node current equations or current equalities using Kirchhoff's Law (they depend on s)

- You need N equations for N unknowns

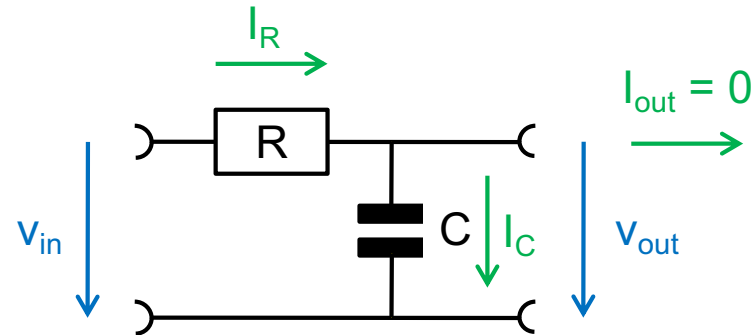
- Solve for the quantity you are interested in (most often V_{out})

- Analyze the result (amplitude / phase / ...)



Example: Low Pass (without load)

- Consider



- We have only *one* unknown: v_{out}

- Current equality at node v_{out} :
$$\frac{V_{in} - V_{out}}{R} = I_R = I_C = v_{out} s C$$

- Solve for v_{out} :
$$\begin{aligned} V_{in} - V_{out} &= v_{out} s C R \\ V_{in} &= v_{out} (1 + s C R) \\ \frac{V_{out}}{V_{in}} &= H(s) = \frac{1}{1 + s C R} \end{aligned}$$



Mathematica Hint

- Write down each node equation (here only one):

$$\text{EQ1} = \frac{v_{in} - v_{out}}{R} == v_{out} s C;$$

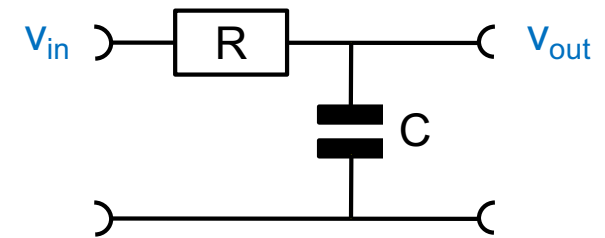
- Solve them:

```
Solve[EQ1, vout] // First
```

$$\left\{ v_{out} \rightarrow \frac{v_{in}}{1 + C R s} \right\}$$

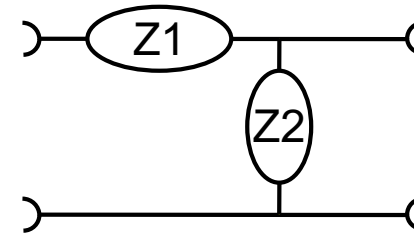
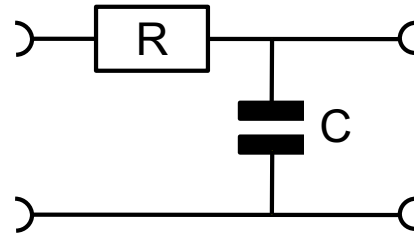
- Define a transfer function:

$$H[s_] = \frac{v_{out}}{v_{in}} /. \% \\ \frac{1}{1 + C R s}$$





Low Pass as 'complex' voltage divider



- The LowPass can be seen as a 'AC' voltage divider with two impedances $Z_1 = R$ and $Z_2 = 1/sC$
- Using the voltage divider formula, we get

$$H(s) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Z_2}{Z_2 + Z_1} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R} = \frac{1}{1 + sRC} = \frac{1}{1 + i\frac{\omega}{\omega_0}}$$

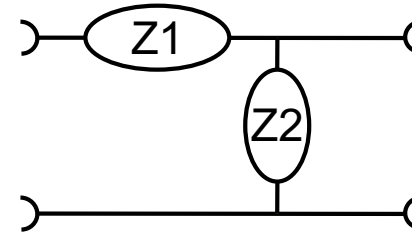
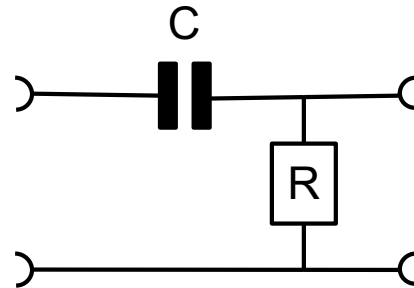
with $\omega_0 = 1/(RC)$, the 'corner frequency'.

- This is the same as before...



The HIGH Pass

- By exchanging R and C, low frequencies are blocked and high frequencies pass through.

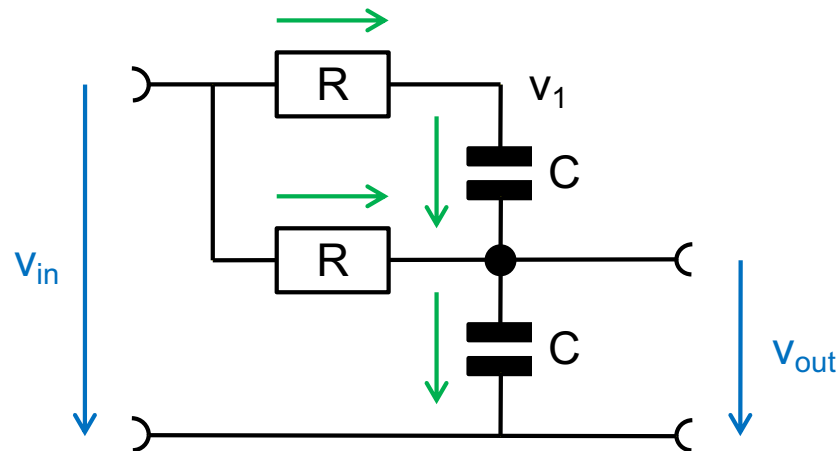


- We get $H_{HP}(s) = \frac{R}{R + \frac{1}{sC}} = \frac{sRC}{1 + sRC}$

- This is the (first order) 'High-Pass'.



A More Complicated Example



- We now have *two unknowns*: v_1 , v_{out}

$$EQ1 (@v_1) : \frac{V_{in} - v_1}{R} = (v_1 - v_{out})sC$$

$$EQ2 (@v_{out}) : (v_1 - v_{out})sC + \frac{V_{in} - v_{out}}{R} = v_{out} sC$$

- *Eliminating v_1 gives:*

$$H(s) = \frac{1 + 2RC s}{1 + 3RC s + (RC)^2 s^2}$$

- This is a *second order TF*. (order = max. exponent of s)

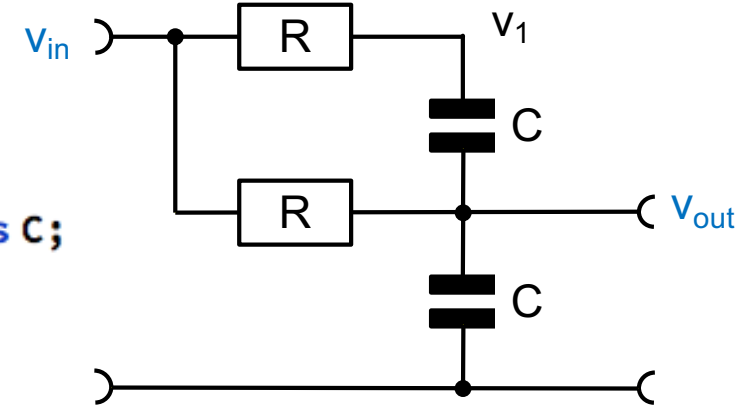


The same using Mathematica

- Node equations (here 2):

$$EQ1 = \frac{v_{in} - v_1}{R} == (v_1 - v_{out}) s C;$$

$$EQ2 = \frac{v_{in} - v_{out}}{R} + (v_1 - v_{out}) s C == v_{out} s C;$$



- Solve them:

`Solve[{EQ1, EQ2}, {vout, v1}] // First`

$$\left\{ v_{out} \rightarrow -\frac{-v_{in} - 2 C R s v_{in}}{1 + 3 C R s + C^2 R^2 s^2}, v_1 \rightarrow \frac{(1 + 3 C R s) v_{in}}{1 + 3 C R s + C^2 R^2 s^2} \right\}$$

- Define a transfer function:

$$H[s_] = \frac{v_{out}}{v_{in}} /. \% // Simplify$$

$$\frac{1 + 2 C R s}{1 + 3 C R s + C^2 R^2 s^2}$$

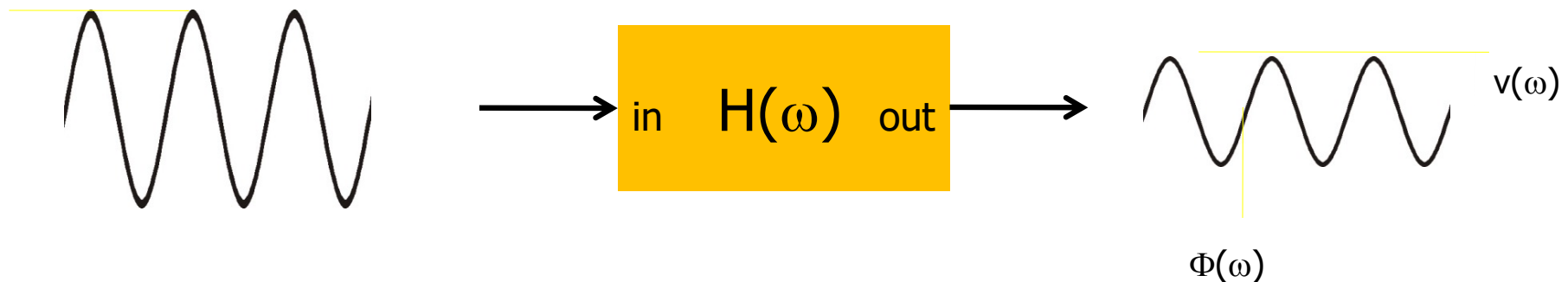


BODE PLOT



Transfer Function (TF)

- The **transfer function** of a *linear, time invariant* system visualizes how the **amplitude** and **phase** of a **sine wave** input signal of **constant frequency** ω appears at the output
- The frequency remains unchanged
- The transfer function $H(\omega)$ contains
 - The phase change $\Phi(\omega)$
 - The gain $v(\omega) = \text{amp_in} / \text{amp_out}(\omega)$





Bode Diagram: Definition

- The Bode Plot shows gain (+ phase) of the transfer function
- The frequency (x-axis) is plotted **logarithmically**
- Gain is plotted (y-axis) **logarithmically**, often in **decibel**

• $DB(x) = 20 \log_{10}(x)$:

$\times 10$ +20 dB

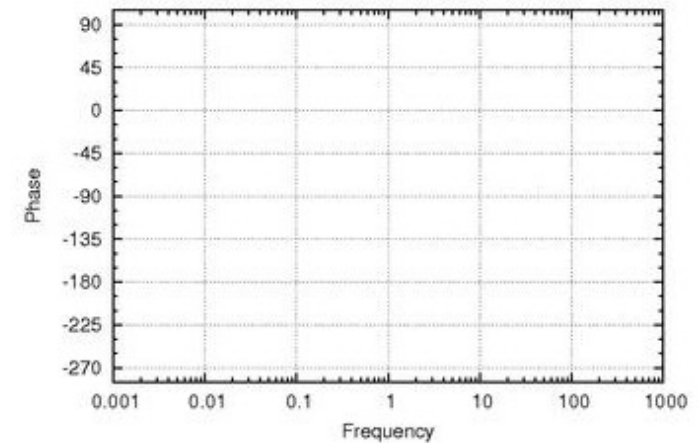
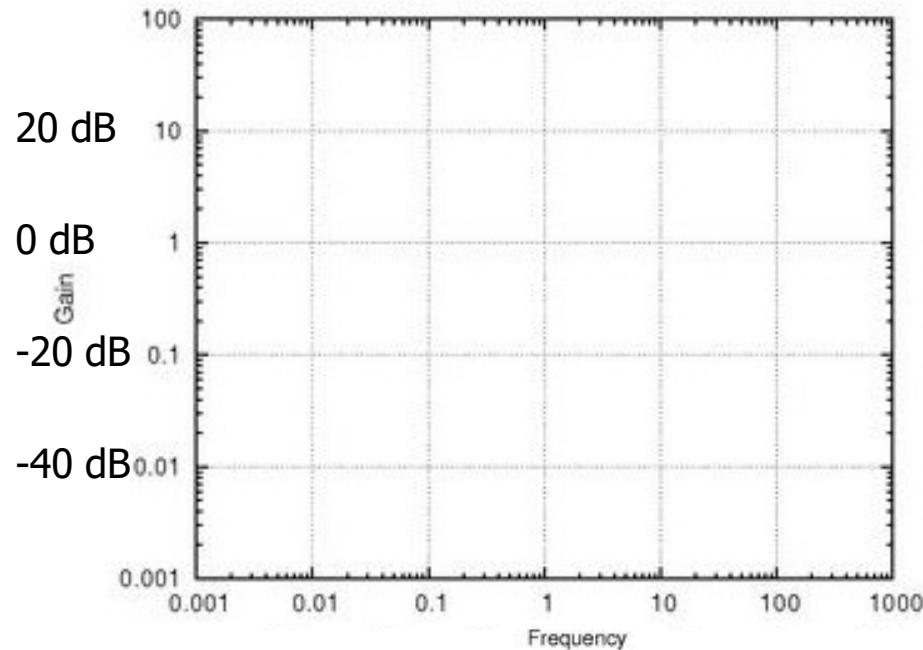
$\times 100$ +40 dB

$\times 2$ 6 dB (not exactly!)

$\times 1$ 0 dB

$/ 2$ -6 dB

$/ \sqrt{2}$ -3 dB



• dBs for multiplied quantities just add !

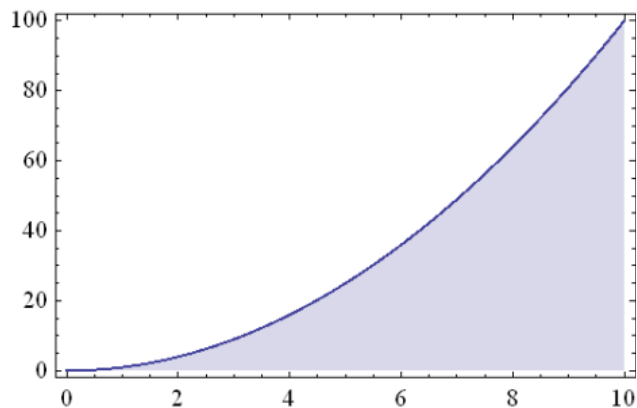


Bode Diagram: Properties

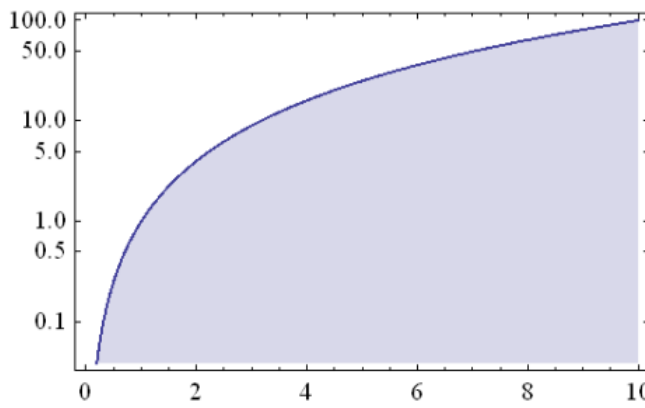
- Power functions are straight lines:

$$f(x) = x^n \Rightarrow \log[f(x)] = n \log(x)$$

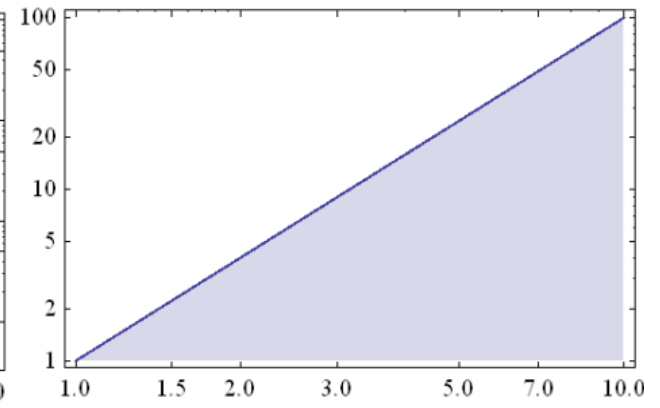
`Plot[x2,{x,0,10}]`



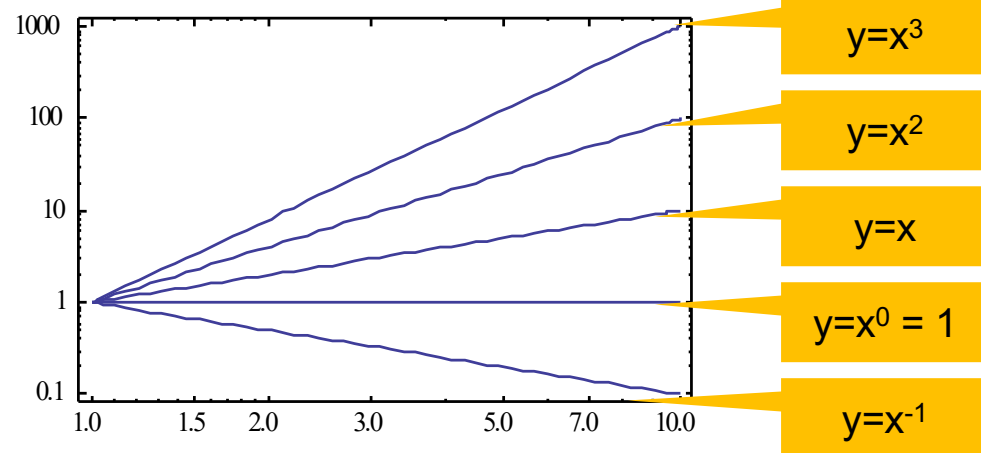
`LogPlot[x2,{x,0,10}]`



`LogLogPlot[x2,{x,1,10}]`



`LogLogPlot[Table[xN,{N,-1,3}],{x,1,10}]`





Bode Diagram: Properties

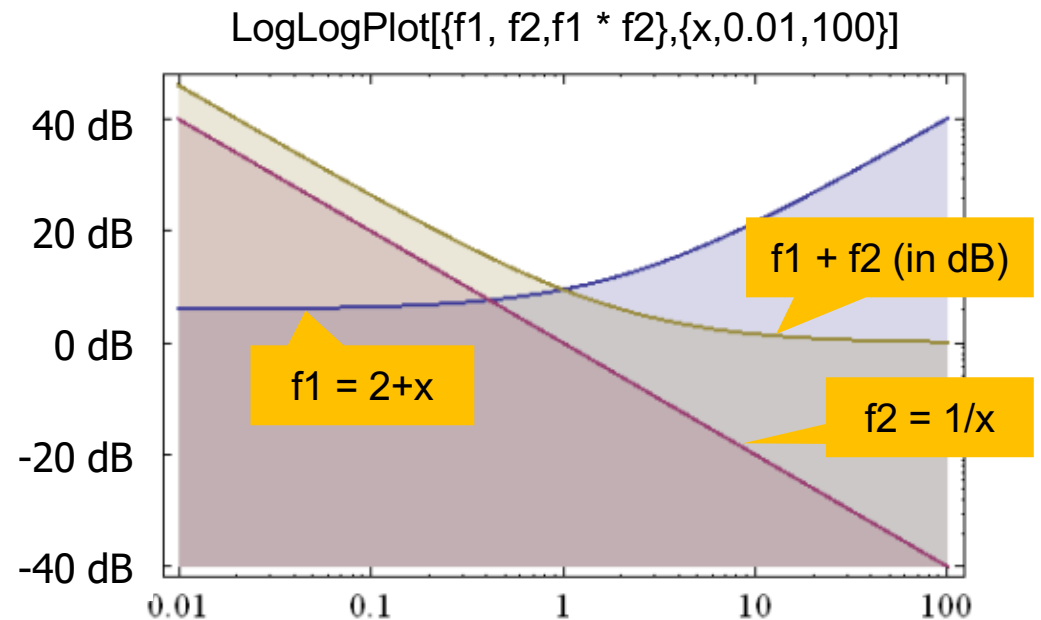
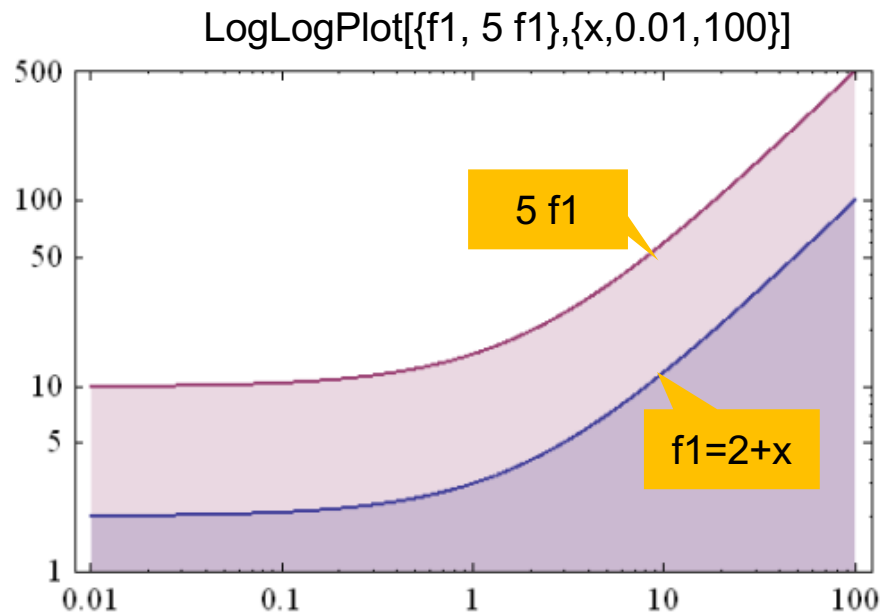
- $1/x$ function has slope -1 :

$$f(x) = \frac{1}{x} = x^{-1} \Rightarrow \log[f(x)] = -1 \log(x)$$

- Multiplied functions are **added** in plot:

$$f = f_1 \cdot f_2 \Rightarrow \log[f] = \log(f_1) + \log(f_2)$$

$f_1=2+x; f_2=x^{-1};$





THE LOW PASS FILTER



Analysis of the Low Pass Transfer Function

▪ **Transfer Function:** $H(\omega) = \frac{1}{1 + i\frac{\omega}{\omega_0}}$ with $\omega_0 = 1/RC$

▪ **Magnitude:** $v(\omega) = \sqrt{H(\omega)H^*(\omega)} = \frac{1}{\sqrt{(1 + i\frac{\omega}{\omega_0})(1 - i\frac{\omega}{\omega_0})}}$

$$v(\omega) = \frac{1}{\sqrt{1 + \frac{\omega^2}{\omega_0^2}}}$$

→ 1 for $\omega \rightarrow 0$

→ $\frac{1}{\sqrt{2}}$ for $\omega = \omega_0$

→ $\frac{\omega_0}{\omega}$ for $\omega \rightarrow \infty$

▪ **Phase:** $H(\omega) = \frac{1}{1 + i\frac{\omega}{\omega_0}} = \frac{1}{1 + i\frac{\omega}{\omega_0}} \times \frac{1 - i\frac{\omega}{\omega_0}}{1 - i\frac{\omega}{\omega_0}} = \frac{1 - i\frac{\omega}{\omega_0}}{1 + \frac{\omega^2}{\omega_0^2}}$

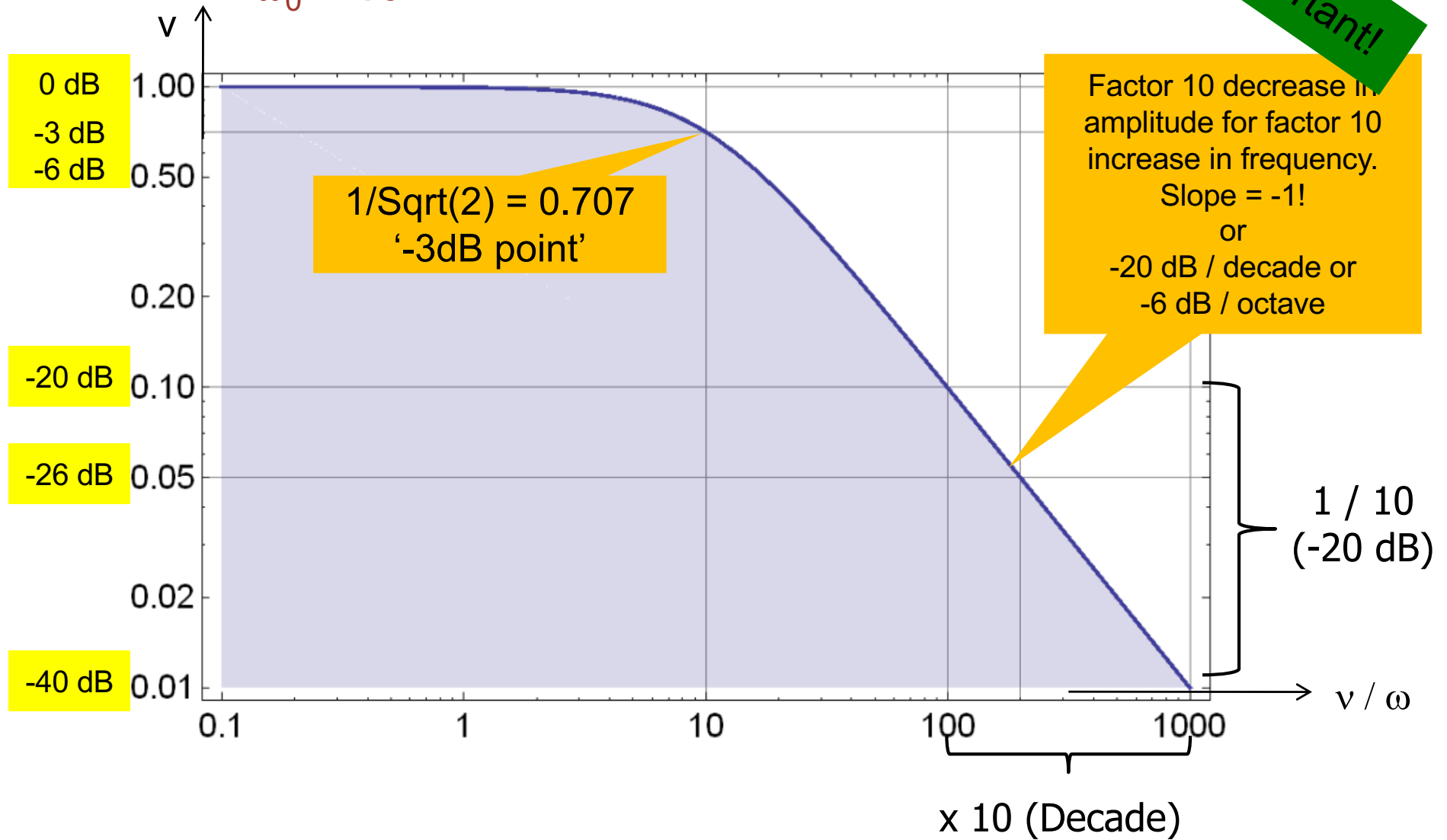
$$\varphi = \text{atan} \left(\frac{\text{Im}(H)}{\text{Re}(H)} \right) = -\text{atan} \left(\frac{\omega}{\omega_0} \right) \quad (\text{rad or degree})$$



Bode Plot of LowPass (Amplitude)

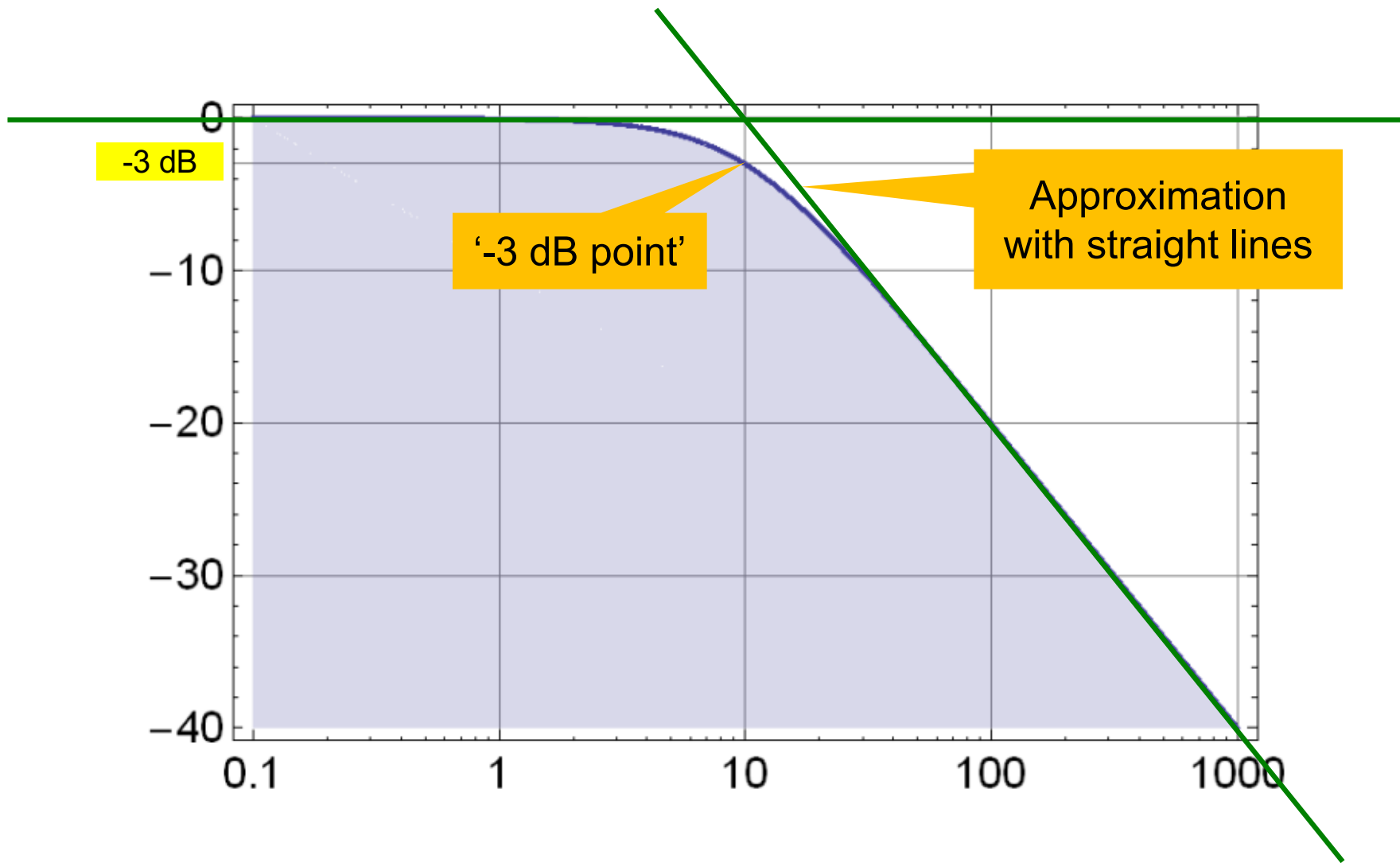
VERY important!

■ $\omega_0 = 10$





The same in dB

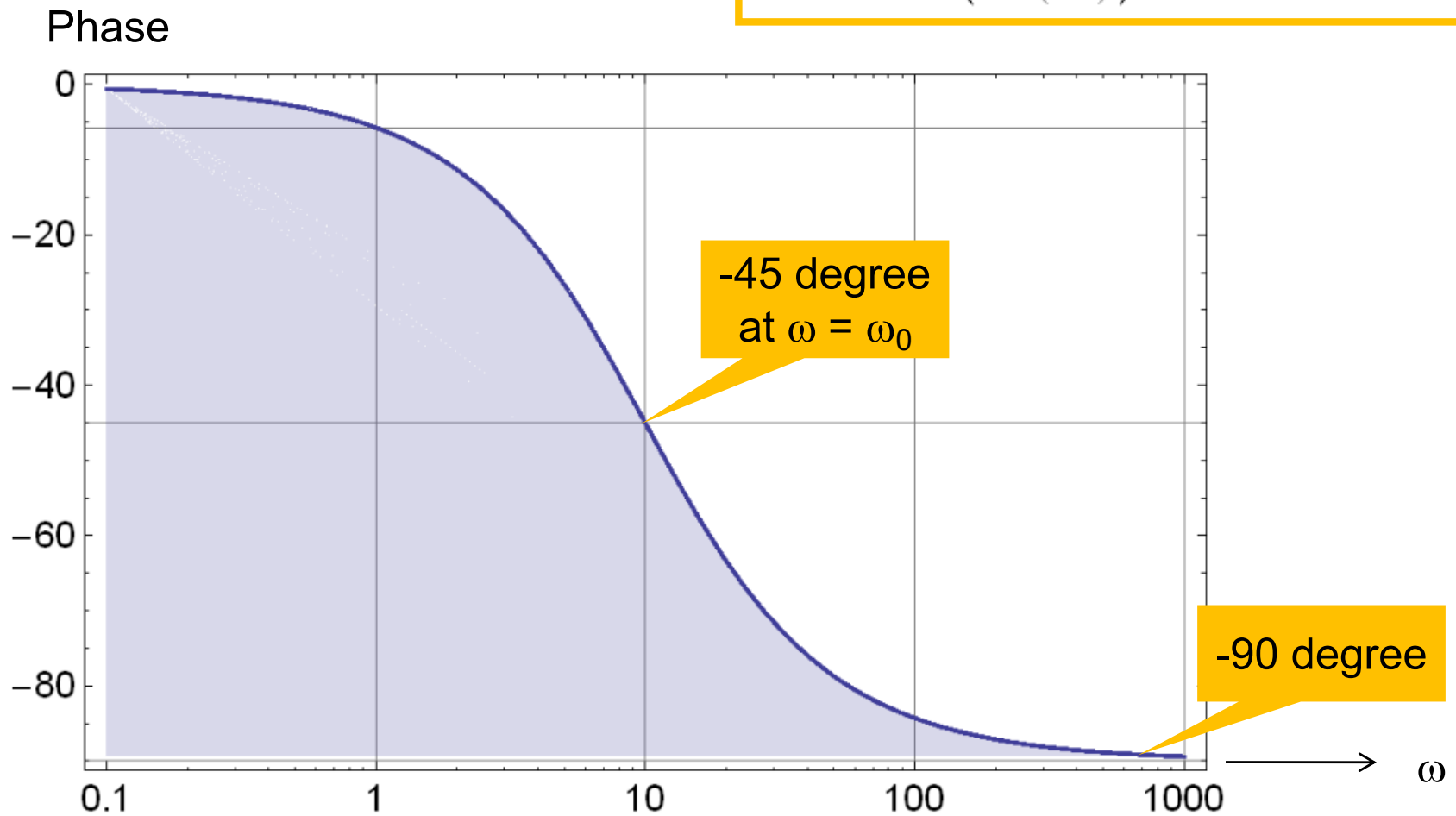




Bode Plot of LowPass (Phase)

- $\omega_0 = 10$
- Lin-Log Plot!

$$\varphi = \text{atan} \left(\frac{\text{Im}(H)}{\text{Re}(H)} \right) = -\text{atan} \left(\frac{\omega}{\omega_0} \right)$$





Where is the Corner?

$$H(\omega) = \frac{1}{1 + i \frac{\omega}{\omega_0}}$$

- At the corner frequency $\omega_0 = 1/(RC)$:
- The impedance of the capacitor is

$$1/(sC) = 1/(i \omega_0 C) = R/i$$

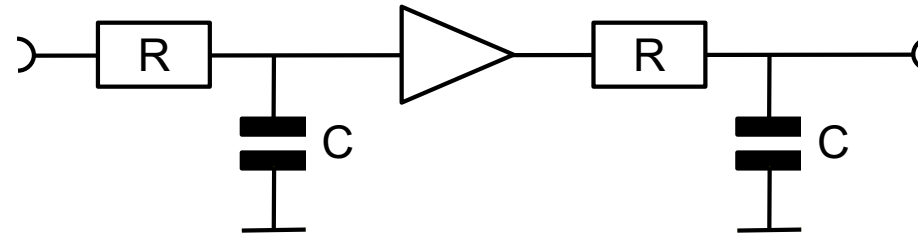
with absolute value R.

- Therefore: At the corner frequency, the (absolute value) of the impedances of the capacitor and the resistor are the same.
 - C becomes 'more important' than R

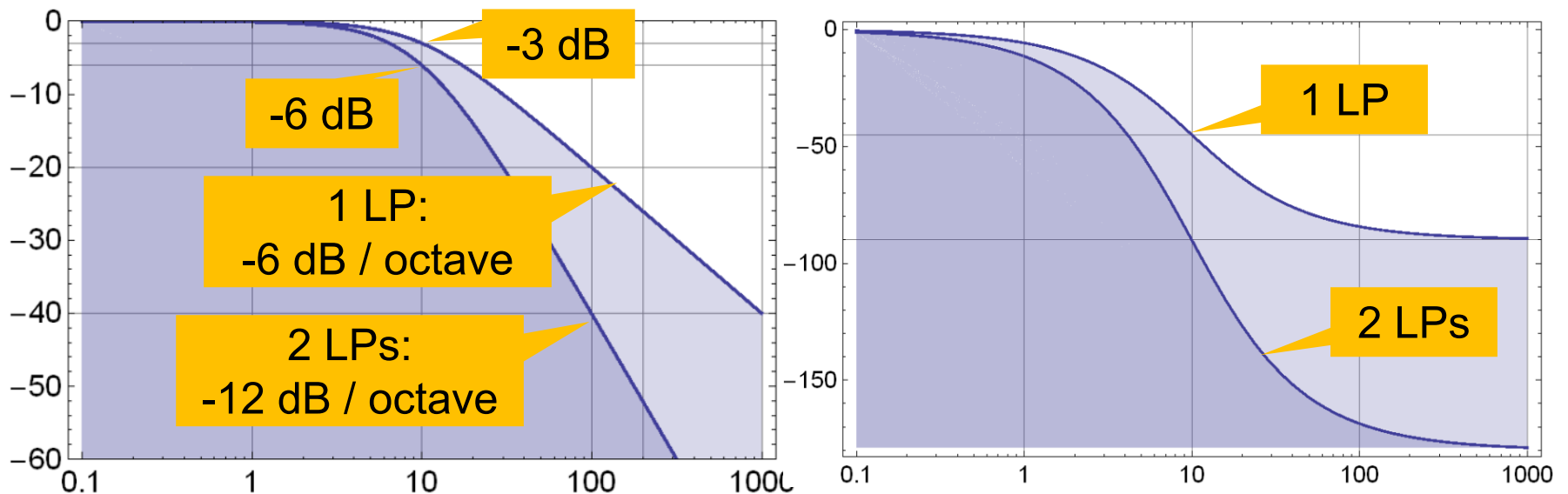


Series Connection of two Low Pass Filters

- Consider two identical LP filters. A 'unit gain buffer' makes sure that the second LP does not load the first one:



- From the properties of the LogLog Plot, the TF of the 2nd order LP is just the sum of two 1st order LPs:





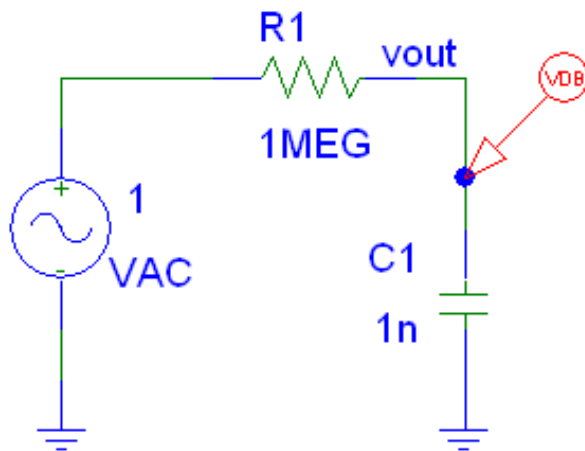
Why bother so much about the low pass ?

- All circuits behave like low-passes (at some frequency)!

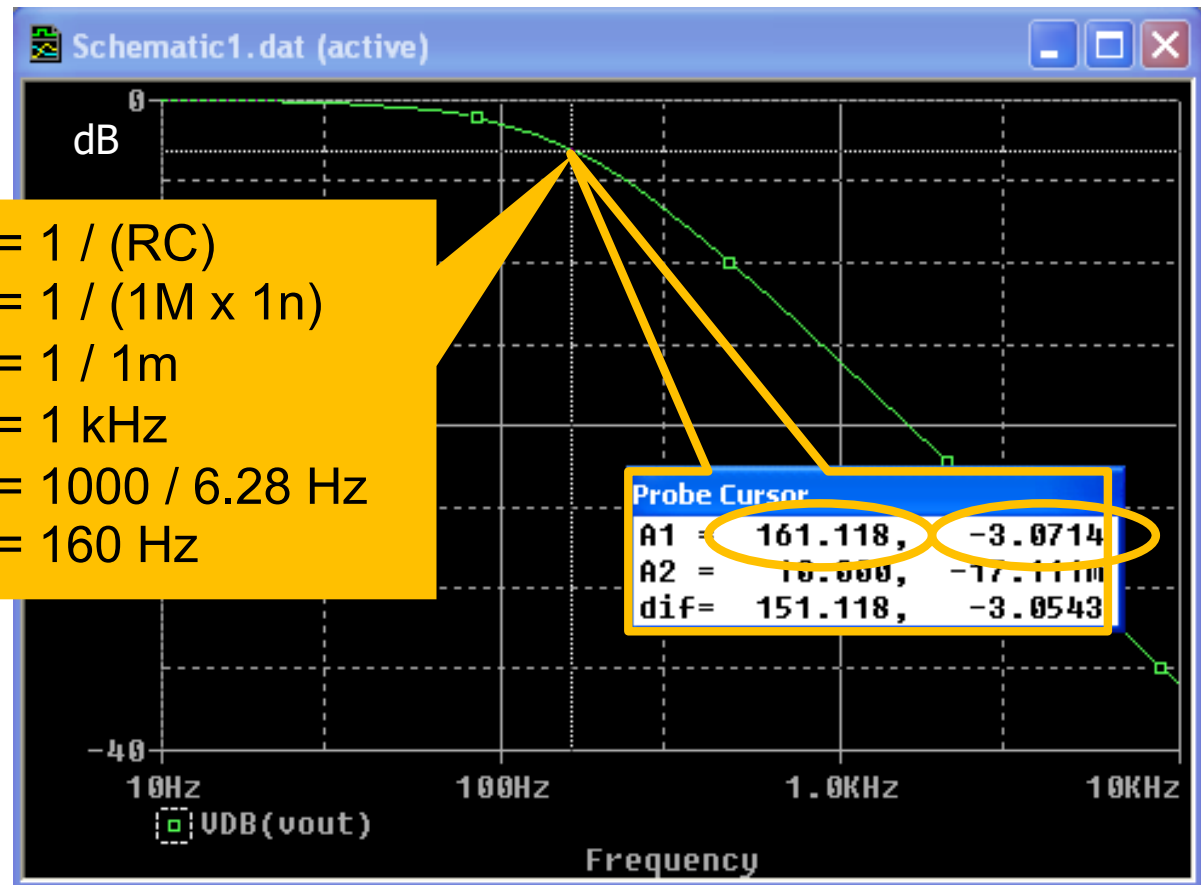


Caveat!

- So far, frequency is expressed with ω , i.e. in radian / second
- We have: $\omega = 2 \pi \nu$
- Therefore, the frequencies in Hertz are 2π lower!!!



$\omega_0 = 1 / (RC)$
 $= 1 / (1M \times 1n)$
 $= 1 / 1m$
 $= 1 \text{ kHz}$
 $\nu = 1000 / 6.28 \text{ Hz}$
 $= 160 \text{ Hz}$





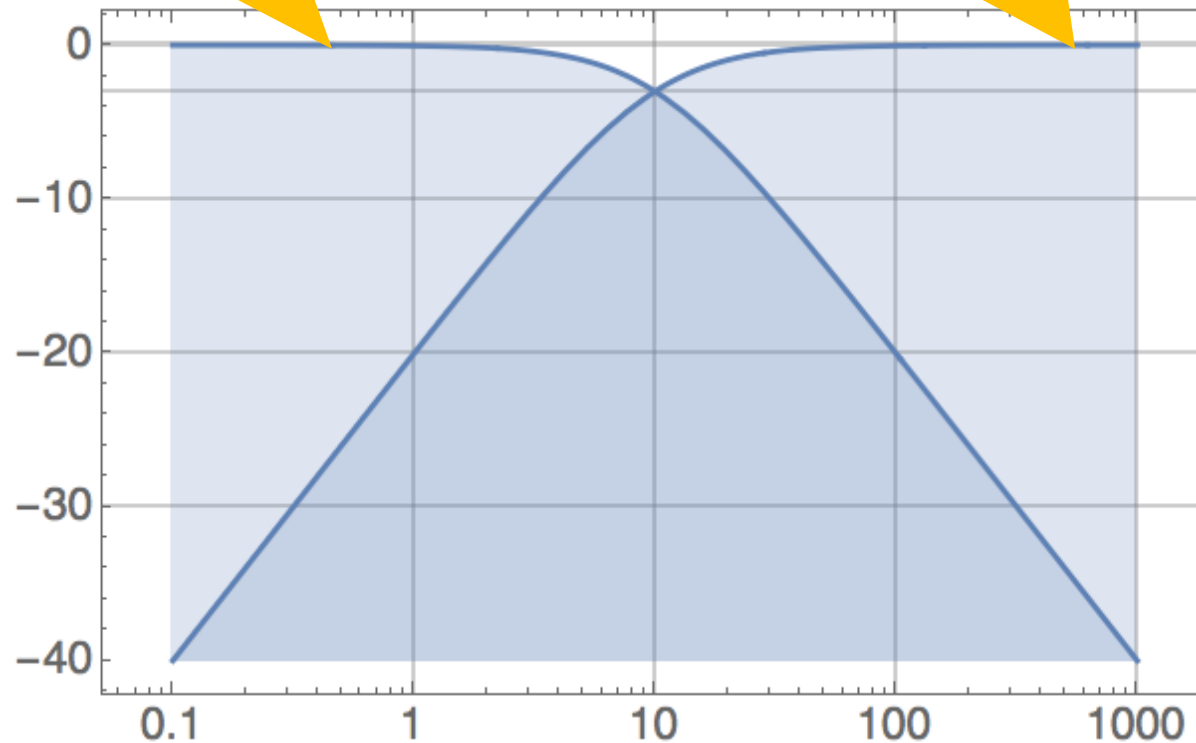
Low Pass and High Pass

$$LP[\omega] = \frac{1}{1 + i \frac{\omega}{\omega_0}}$$

$$HP[\omega] = \frac{i \frac{\omega}{\omega_0}}{1 + i \frac{\omega}{\omega_0}};$$

$$LPgain(\omega) = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}$$

$$HPgain(\omega) = \frac{\frac{\omega}{\omega_0}}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}$$





Bode Plots with Mathematica

- Replace s by $i \omega$
- Calculate (squared) gain as absolute value

```
gain2 = H[i ω] Conjugate[H[i ω]] // ComplexExpand // Simplify
```

$$\frac{1 + 4 C^2 R^2 \omega^2}{1 + 7 C^2 R^2 \omega^2 + C^4 R^4 \omega^4}$$

- To plot, convert to dB by taking $20 \text{Log}_{10}[\sqrt{H}]$.
 - The sqrt can be eliminated by using $10 \text{Log}_{10}[H]$

```
LogLinearPlot[10 Log[10, gain2] /. {R → 1, C → 1}, {ω, 0.01, 100},  
PlotRange → {-20, 2}, Filling → -20]
```

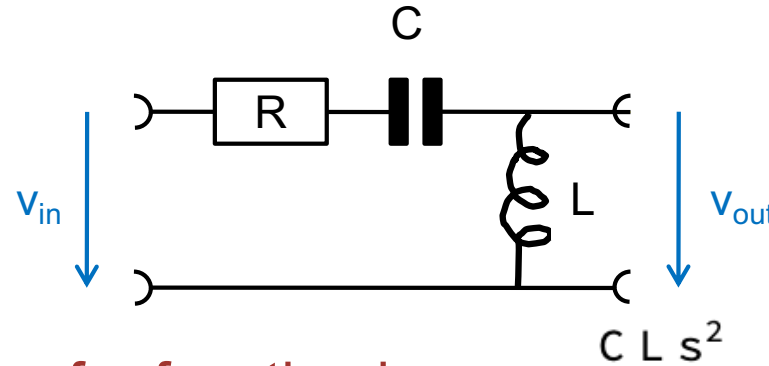
- For phase, better use $\text{ArcTan}[\text{Re}, \text{Im}]$ to get quadrant right

```
LogLinearPlot[ $\frac{180}{\pi}$  ArcTan[Re[H[i ω]], Im[H[i ω]]] /. {R → 1, C → 1}, {ω, 0.01, 100}]
```



A More Complex Example

- Consider a (High Pass) filter with an inductor:

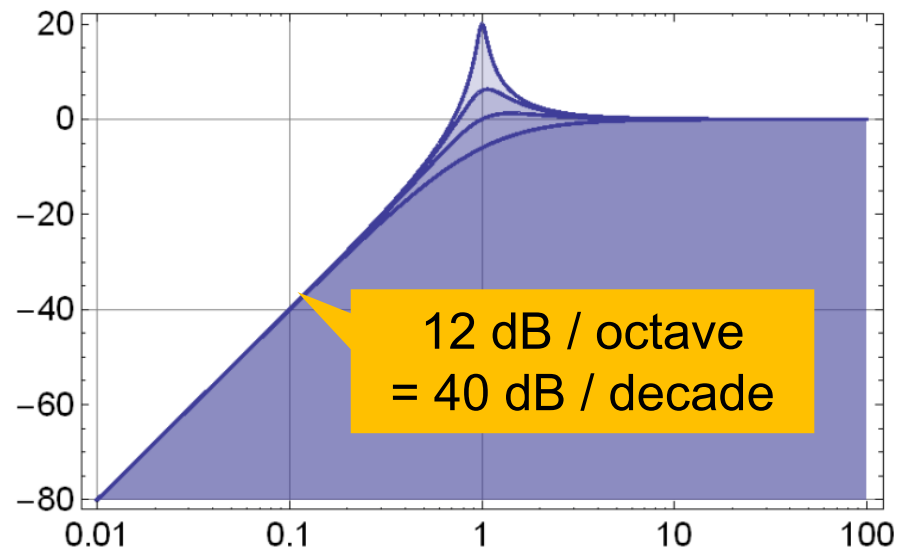


Mathematica
Demo

- The transfer function is $\frac{C L s^2}{1 + C R s + C L s^2}$
- It is of 'second order' (s has exponent of 2 in denominator)

- Magnitude:
L=C=1
R=0.1,0.5,1,2

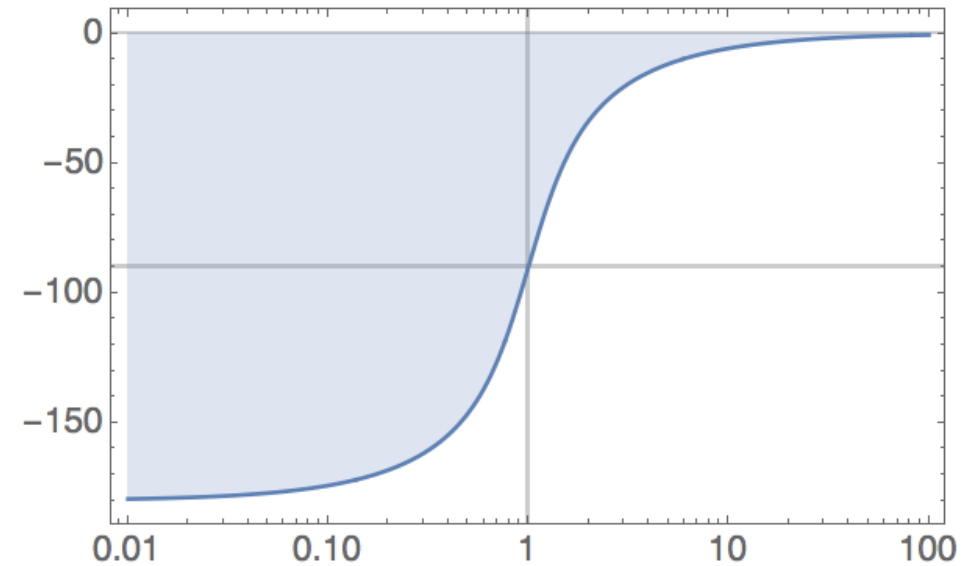
- 'Resonance'
- 'Inductive peaking'





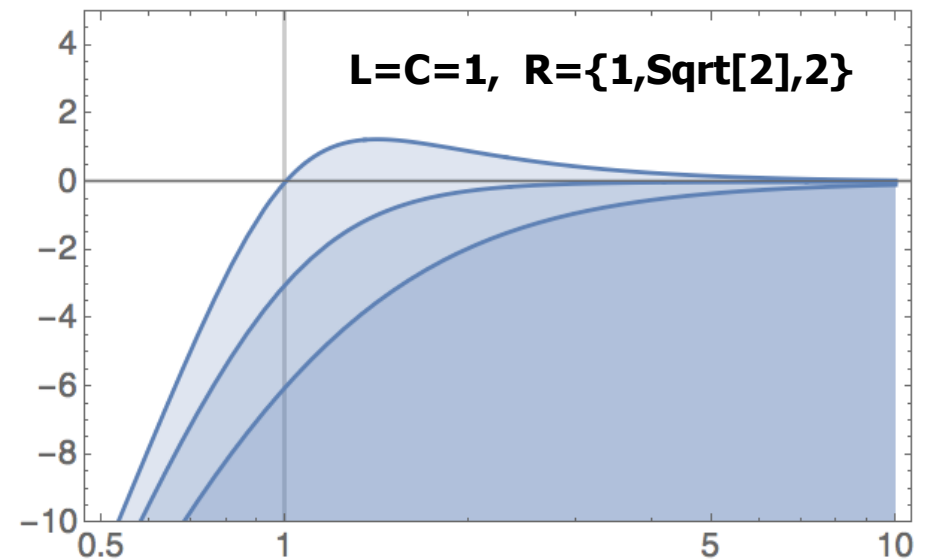
Phase

■ Phase



■ For fun:

- When is filter steep & flat?
- Zoom to corner frequency:



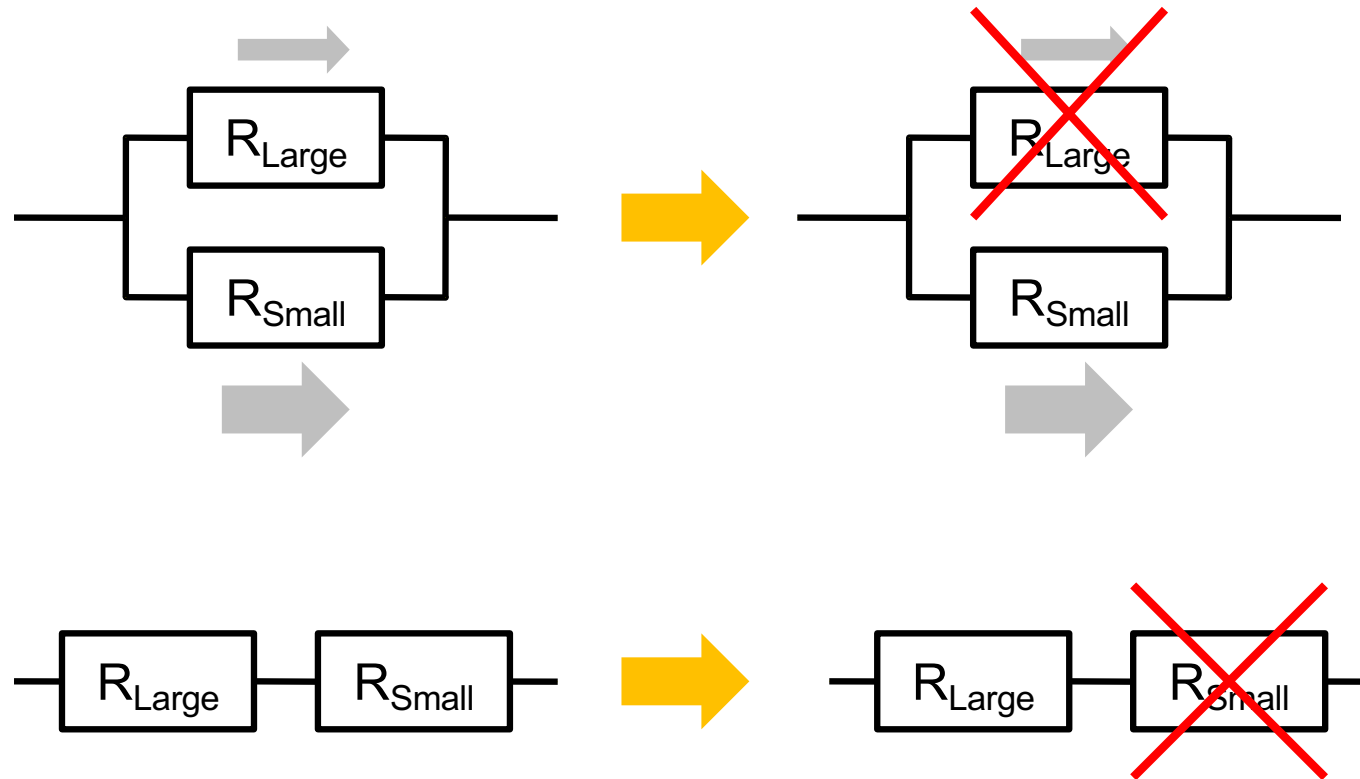


CIRCUIT SIMPLIFICATIONS



Large and Small Values

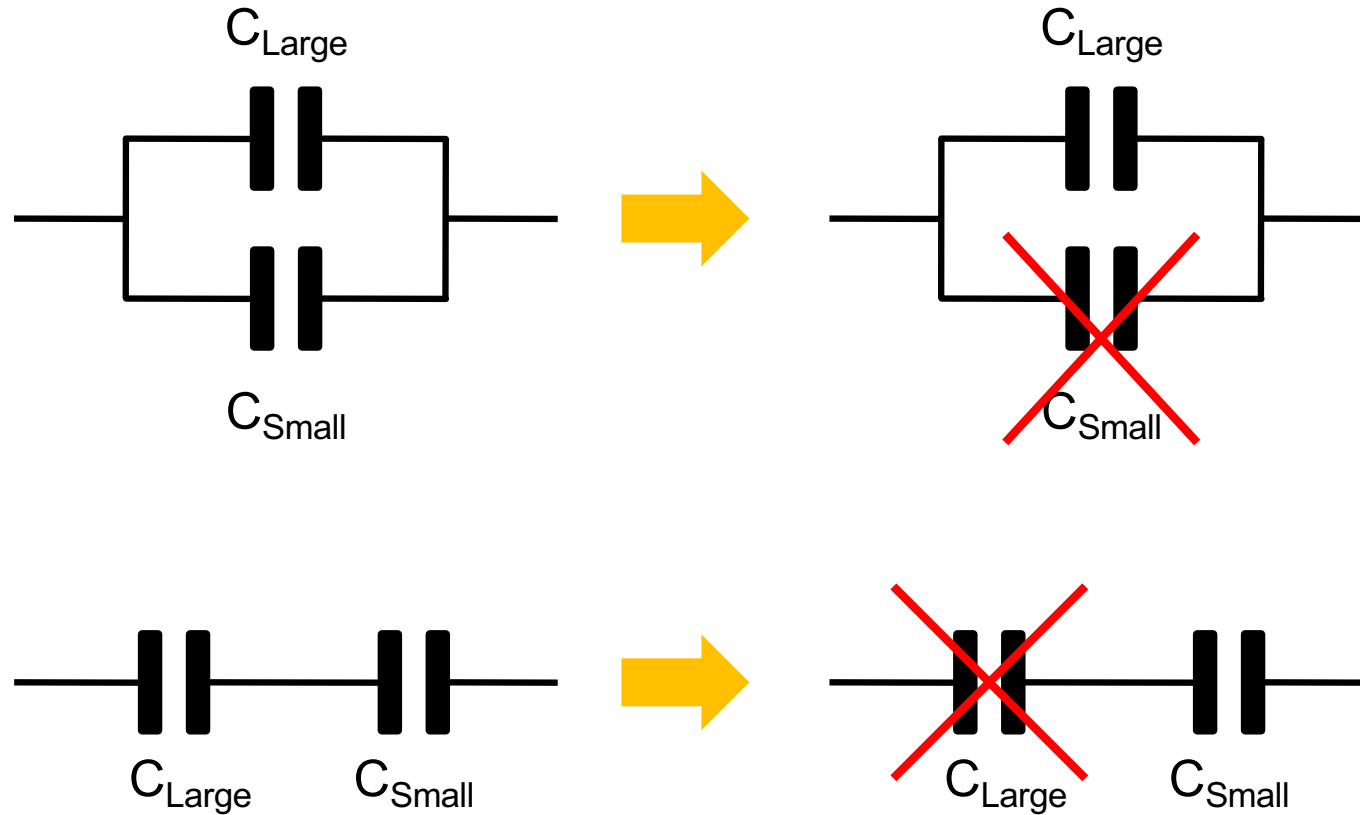
- To roughly understand behavior of circuits, only keep the dominant components:



- Eliminate *larger* or the *smaller* part (depending on circuit!)
- Error \sim ratio of components



The same for Capacitors





Resistors AND Capacitors

- Behavior depends on frequency ($|Z_C| = 1/(2\pi\nu C)$)

